MMP Learning Seminar

Week 63:

Introduction to birational boundedness of varieties of log general type.

Boundedness of varieties of log general type:

Curves: C curve with ω_C ample $\Longleftrightarrow g(C) \geq 2$.

$H^0(\omega_X)$ induces a morphism $C \xrightarrow{\phi_1} \mathbb{P}^{g-1}$.

However, in general this is not an embedding, e.g. hyperelliptic

$\omega_X^{\otimes 3}$ is very ample; it separates points and tangent dir.

$H^0(\omega_X^{\otimes 3})$ defines an embedding

$$C \xrightarrow{\phi_3} \mathbb{P}^N = \mathbb{P} H^0(X, \omega_X^{\otimes 3}).$$

$$N+1 = h^0(X, \omega_X^{\otimes 3}) = \chi(X, \omega_X^{\otimes 3}) = 3\deg \omega_X + \chi(X, \mathcal{O}_X)$$

$$= 5\chi(X, \omega_X) = 5g-5.$$

$$\deg(\omega_X^{\otimes 3}) = 6g-6.$$

All curves of genus g can be embedded in $\mathbb{P}^{5g-6}$ as curves of degree $6g-6$.

Surface: X smooth surface with $\underline{\omega_X \text{ ample}}$.

$$\mathrm{Vol}(X) = K_X^2. \qquad \text{canonically polarized.}$$

Theorem: Canonically polarized surfaces with fixed volume K_X^2 form a bounded family.

$\omega_X^{\otimes 5}$ defines an embedding into $\mathbb{P}^N$ of degree $25\,\mathrm{vol}(X)$.

Aim: Given the family of all smooth projective canonically polarized varieties of volume $= d$ & dimension $= n$ $\mathcal{F}_{n,d}$.

Show that there exists a constant $m = m(d,n)$ s.t.

$\omega_X^{\otimes m}$ defines an embedding of X into a projective space.

Lifting sections from lower dimensions:

$X \in \mathfrak{F}_{n,d}$, $\quad H \in |mK_X|$ we can consider $t = \mathrm{lct}(X; H)$.
$\quad\quad\quad\quad\quad\quad \geq 0$

$K_X + tH \equiv (tm+1)K_X$ is strictly log canonical.

Z be a minimal lcc of the pair (X, tH).

Using some vanishing Theorems (e.g. Nadel), we prove $H^1(K_X + (t-1)H) = 0$.

so we will get a surjective homomorphism:

the pair obtained by adj.

$$H^0(K_X + tH) \longrightarrow H^0(K_Z + H_Z).$$

Remark 1: We need to control the sing of (Z, H_Z), and we know these are semi log canonical (slc varieties)

Remark 2: Is often the case that we need to control the coeff of H_Z, For instance, put them in a set satisfying the DCC.

(DCC coefficients).

We need to enlarge the set $\mathfrak{F}_{n,d}$.

Boundedness of varieties of log general type:

We work over an algebraically closed field of charact zero.

Theorem 1.1: Fix an integer n, a positive rational number d, and a set $I \subseteq [0,1]$ which satisfies the DCC.

Then, the set $\mathfrak{F}_{slc}(n,d,I)$ of all the pairs (X,Δ) such that:

(1) X is projective of dimension n,

(2) (X,Δ) is semi log canonical,

(3) the coefficients of Δ belong to I,

(4) $K_X+\Delta$ is an ample $\mathbb{Q}$-divisor, and

(5) $(K_X+\Delta)^n = d$.

in the ACC paper we proved this under the hypothesis that (X,Δ) are klt & $\mathrm{ld}(X,\Delta) \geq \varepsilon > 0$

is bounded.

In particular, there exists a finite set I_0 such that

$\mathfrak{F}_{slc}(n,d,I) = \mathfrak{F}_{slc}(n,d,I_0)$.

Abundance in families:

Theorem 1.2: Suppose that (X,Δ) is a log pair and the coefficients of Δ are in $[0,1]\cap\mathbb{Q}$. Let $\pi: X \longrightarrow U$ be a projective morphism to a smooth variety such that (X,Δ) is log smooth over U.

If there is a closed point $0\in U$ such that the fiber (X_0,Δ_0) has a good minimal model, then every fiber has a good minimal model.

Canonically polarized varieties:

Step 1: Reduce to log canonical pairs.

(X,Δ) semi log canonical

$n: Y \longrightarrow X, \qquad K_Y + \Gamma = n^*(K_X+\Delta)$

$\Gamma :=$ reduced conductor + strict transform of Δ.
$\qquad\qquad\qquad \| $
$\qquad\qquad\qquad S$

(X,Δ) is determined by (Y,Γ) and an involution $\tau: S \to S$.

$\mathcal{L}$ be the pull-back of $m(K_X+\Delta)$. is very ample

τ fixes $\mathcal{L}$., τ embeds inside a linear alg group

$$\mathbb{P}GL\, H^0(Y,\mathcal{L}).$$

This argument implies that is enough to work with lc pairs.

Canonically polarized varieties :

Step 2: We need to control the number of irreducible components of X.

$\mathrm{vol}(X, K_X+\Delta) = d.$

$X = C$ slc curve of genus g. then K_C has degree $2g-2$, so X has at most $2g-2$ components.

$X = \cup X_i$, $K_X+\Delta|_{X_i} = K_{X_i}+\Delta_i$, we can show:

$$d = \mathrm{vol}(K_X+\Delta) \geq \sum_i \mathrm{vol}(\underbrace{K_{X_i}+\Delta_i}_{\text{slc.}})$$

Theorem: Fix n a positive integer & $I \subseteq [0,1]$ satisfying the DCC. Let $\mathfrak{D}$ be the set of all log canonical pairs (X,Δ) s.t. the dimension of X is n & the coefficients of Δ belong to I. Then, the set

$$\{ \mathrm{vol}(X, K_X+\Delta) \mid (X,\Delta) \in \mathfrak{D} \}$$

also satisfy the descending chain condition.

Canonically polarized varieties:

Step 3: Reduce to a horizontal family over a finite type base

$\mathfrak{F} \subseteq \mathfrak{F}_{slc}(n, d, I)$ be the subset of irreducible log canonical pairs.

$\mathfrak{F}$ is log birationally bounded:

$$\begin{array}{c} (Z, B) \\ \downarrow \\ U \end{array}$$

$(X, \Delta) \in \mathfrak{F}$, we can find $u \in U$,

$Z_u \overset{\phi}{\dashrightarrow} X$ such that

$\operatorname{supp}(\phi_*^{-1}\Delta + Ex(\phi)) \subseteq \operatorname{Supp} B_u$.

proved in one of the previous HMX papers.

We aim to show that $\mathfrak{F}$ is indeed bounded

Example: U_0 configuration of k lines in $\mathbb{P}^2$.

$Z_0 = \mathbb{P}^2 \times U_0$, $B_0 =$ reduced div of the lines

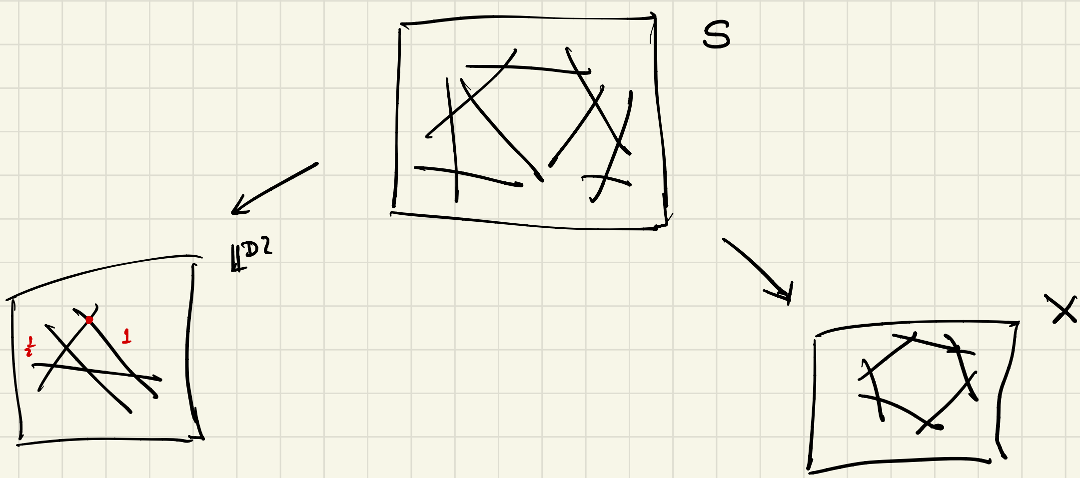
Take $I_0 = \{\frac{1}{2}, 1\}$.

ex(ϕ) + strict transform of Δ.

We would like the volume of $\operatorname{vol}(Z_u, K_{Z_u} + \$) = d$ be cst.

How do we produce an element of $\mathcal{F}$?

Example: $(X,\Delta) \in \mathcal{F}$ is constructed by blowing up points of $(\mathbb{P}^2, \lambda_1 L_1 + \dots + \lambda_k L_k)$, $\lambda_i \in \{\frac{1}{2}, 1\}$ until we arrive to a model S and then we blow down (-1)-curves until we arrive to X.



The pair (S,Γ) is again of log general type, & we can choose (S,Γ) so that it has the same volume as (X,Δ).

We may assume that all the blow-ups are horizontal over U.

$(Z, B) \longrightarrow U$ & possible blow-ups.

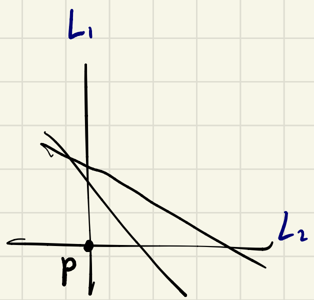
Step 4: Reduce to the case in which

$$(Z_u, K_{Z_u} + \Phi) \quad \text{and} \quad (X, \Delta)$$

have the same volume:

Example: $(\mathbb{P}^2, L_1 + L_2 + L_3 + L_4) = Z$

& X are the possible blow-ups with fixed volume.



perform a weighted blow-up at the point p with weights (a,b).

$$f : X \longrightarrow \mathbb{P}^2.$$

$$K_X + M_1 + M_2 + M_3 + E \sim 0$$

$$(K_X + M_1 + \cdots + M_4)^2 = (M_4 - E)^2 = M_4^2 + E^2 = E^2 + 1.$$

Toric geometry: $E^2 = -1/ab$.

New volume is $1 - \frac{1}{ab} = d$ fixed,

then (a,b) has only finitely many possible values.

Step 5: Deduce that the (X,Δ)'s are in a bounded family.

$$\begin{array}{ccl} (Z,B) & \dashrightarrow & (X,\Delta) \\ \big\downarrow & & \\ U. & & \end{array}$$

, they have the same volume & $K_X+\Delta$ is ample

So in other words, $K_X+\Delta$ is an ample model of (Z,B).

Aim: Show that the set of fibers with ample models is constructible.

Remark: If (Z,B) is klt, then BCHM tell us this is the case.

Under the conditions of the statement, we can use 1.2. to show that there is one fiber with ample model implies that all fibers have models. $(*)$.

We will obtain that all fibers admit ample models, so the set of ample models $(X,\Delta)\in\mathcal{F}$ belong to a bounded family..